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# A resummation of large sub-leading corrections at small $x$.

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Abstract: The NLL corrections to the BFKL kernel are known to be very large, to the extent that even for small values of $\alpha_s$, they lead to physical cross sections which are not positive definite. It is shown in the context of a toy model, that such pathological behaviour is an artifact of the truncation at NLL order, and is associated in particular with double transverse logarithms. These are resummed in a manner consistent with the full NLL kernel, and are shown to change its properties quite considerably.

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## Contents



# 1. Introduction

Following many years of calculations [1] the full NLL corrections to the BFKL kernel [2] recently became available [3, 4]. Quite soon afterwards it was pointed out that there were serious problems with the convergence of the kernel [5], and in particular that for values of $\alpha_s$ above about 0.05 the saddle-point structure of the characteristic function changes: instead of having a single saddle-point on the real $\gamma$ axis, one has two complex-conjugate saddle points [6]. As a result the power of the small-$x$ growth, rather than depending on $\alpha_s$ (schematically) as $\bar{\alpha}_s 4 \ln 2 - N\bar{\alpha}_s^2$, where $N$ is a number, seems to acquire a more complex (non-parabolic) dependence on $\alpha_s$. A particularly worrying consequence of the saddle points having complex values is that the solutions oscillate as a function of transverse momentum (this is a statement which is valid beyond the saddle-point approximation), and even when integrated with the appropriate matrix-elements can lead to negative results for physical cross sections [7].

This paper will consider this pathological behaviour in the context of yet higher order corrections. The discussion will be fundamentally related to the issue of the choice of scales (discussed also in some detail in [4]). In determining the leading contribution to the cross section for the high-energy scattering of two objects with transverse scales

$k_1^2$ and $k_2^2$, one resums terms of the form

$$\left(\alpha_s \ln \frac{s}{s_0}\right)^n .$$

At leading-logarithmic (LL) (in $x$) order, the choice of $s_0$ is immaterial. When one starts to consider next-to-leading-logarithmic (NLL) terms,

$$\alpha_s \left(\alpha_s \ln \frac{s}{s_0}\right)^n ,$$

one sees that their form will depend on the choice made for $s_0$ in the LL terms. Various choices come to mind for $s_0$. A symmetrical $s_0 = k_1 k_2$ is a Regge-motivated choice. But in situations where $k_1^2 \gg k_2^2$ (or $k_2^2 \gg k_1^2$) it is more natural to take $s_0 = k_1^2$ ($s_0 = k_2^2$), because that is the scale which enters in the appropriate DGLAP-type resummation [8]. In this limit the cross section has terms of the form

$$\left(\alpha_s \ln \frac{k_1^2}{k_2^2} \ln \frac{s}{k_1^2}\right)^n .$$

One sees immediately that trying to rewrite them in terms of $s_0 = k_1 k_2$ leads to double transverse logarithms in the series,

$$\left(\alpha_s \ln^2 \frac{k_1^2}{k_2^2}\right)^{n-m} \left(\alpha_s \ln \frac{k_1^2}{k_2^2} \ln \frac{s}{k_1 k_2}\right)^m .$$

Though formally subleading, these terms form a significant part of the NLL corrections to the BFKL kernel, because the double logarithm can be large even when $|\ln k_1^2/k_2^2| \ll \ln s/s_0$.

To study such effects in more detail it is convenient to go the Mellin-transform space of the gluon's Green function (assuming straightforward exponentiation, in the notation of [4]),

$$\int \frac{d\gamma}{2\pi i} \frac{d\omega}{2\pi i} \left(\frac{s}{s_0}\right)^\omega g_\omega(\gamma) \frac{1}{k_1^2} \left(\frac{k_1^2}{k_2^2}\right)^\gamma , \quad g_\omega(\gamma) = \frac{F(\gamma)}{1 - \frac{\bar{\alpha}_s}{\omega}\chi(\gamma)} , \tag{1.1}$$

with $\bar{\alpha}_s = \alpha_s C_A/\pi$, $F(\gamma)$ a combination of $\omega$-independent kernels, and $\chi(\gamma)$ known as the characteristic function. At LL order [2],

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) . \tag{1.2}$$

The effect of the changing $s_0$ from $k_1^2$ to $k_1 k_2$ is to take $\gamma$ to $\gamma + \frac{1}{2}\omega$. As has been discussed by both Fadin and Lipatov [3] and Ciafaloni and Camici [4], such a transformation, acting on the LL kernel, produces a term

$$-\frac{\bar{\alpha}_s}{2\gamma^3} ,$$

which is precisely as found in the NLL kernel (with scale $s_0 = k_1 k_2$). It turns out that this, and the corresponding $\bar{\alpha}_s/2(1-\gamma)^3$ term (which are the first of the double-logarithmic terms discussed above), are responsible for about half the NLL correction to the asymptotic BFKL exponent, $\bar{\alpha}_s\chi(\frac{1}{2})$.

The purpose of this article is to resum the double logarithmic terms. The basis of the discussion will be that with a scale $s_0 = k_1^2$ ($k_2^2$) the appropriate DGLAP limit $\gamma \to 0$ ($1-\gamma \to 0$) must be free of double logarithms, i.e. there should be no terms $\alpha_s^n \gamma^{-k}$ ($\alpha_s^n (1-\gamma)^{-k}$) with $k > n+1$.

In section 2, we will consider a toy kernel, a variant of the linked-dipole-chain (LDC) model [9], which will have the feature of reproducing the BFKL kernel at LL order, and of resumming all the "leading" double-logarithmic (DL) terms, $\alpha_s^n/\gamma^{2n+1}$. Its main interest will be related to the study of the general analytic structure and the order-by-order (non-)convergence of the kernel. It will be shown that the feature noted by Ross [6] at NLL order, namely that the LL saddle point splits into two symmetrical saddle points which no longer lie on the real $\gamma$ axis, is reproduced by this model. This happens at a value of $\alpha_s$ considerably smaller than the point at which the NLL corrections cause the asymptotic exponent to become negative — essentially the NLL corrections have a much larger effect on the second derivative, $\chi''(\frac{1}{2})$ than on $\chi(\frac{1}{2})$ itself.

If one resums the toy kernel, one returns to a single saddle point on the real axis. This is reassuring since the presence of saddle-points off the real axis would almost certainly have led to oscillating physical cross sections [6, 7]. It is to be noted that for reasonable values of $\alpha_s$ ($> 0.1$) it is *n*ot sufficient simply to go to NNLL or in general N$^n$LL order to obtain a well-behaved kernel: the pathologies seen in the NLL kernel persist for any fixed $n$, hence the resummation of an appropriate subset of the N$^n$LL-order terms, for all $n$, is essential.

Section 3 will discuss a way, given a fixed order kernel up to order $\alpha_s^N$, to resum terms

$$\frac{\alpha_s^n}{\gamma^k}, \frac{\alpha_s^n}{(1-\gamma)^k} \qquad 2n+1-N \le k \le 2n+1\,.$$

The technique will be applied to the NLL kernel (with details given in the appendix). The procedure is not unique, in that it introduces subleading terms, with $n > N$, $k < 2n+1-N$ which are not under control. The study of four schemes which differ in their treatment of these subleading terms suggests that the value of $\chi(\frac{1}{2})$ is reasonably stable, i.e. scheme-independent after the resummation. On the other hand $\chi''(\frac{1}{2})$ remains quite unstable, being sensitive in particular to one's treatment of terms $\alpha_s^n/\gamma^{n+1}$.

## 2. An LDC-like toy kernel

The question of how to construct a small-$x$ kernel which reproduces the DGLAP double-logarithmic ($\ln x \ln Q^2$) limit both for $k_1 \gg k_2$ and for $k_1 \ll k_2$ was considered a few years ago by Andersson, Gustafson and Samuelsson [9]. One of the equations that they proposed for the unintegrated gluon distribution $\mathcal{F}$ was[1]

$$\frac{d\mathcal{F}(x,k^2)}{d\ln(1/x)} = \frac{\bar{\alpha}_\mathrm{s}}{\pi}\int \frac{d^2q}{\pi q^2}\left(\mathcal{F}(x',|k+q|^2) - \mathcal{F}(x,k)\,\Theta(k-q)\right) . \tag{2.1}$$

Here, in $\mathcal{F}(x,k_1^2)$, $x$ is defined as $k_1^2/s$, so that we are using $s_0 = k_1^2$. The difference between this and the BFKL equation lies in the presence of $x' = \max(x, x|k+q|^2/k^2)$ in the right hand side. The justification is that in a situation where $|k+q|^2 \gg k^2$, from the DGLAP point of view, $x$ is no longer the appropriate evolution variable, but rather (within the double-logarithmic approximation) $x \cdot |k+q|^2/k^2$.

Taking the Mellin transform one sees that the characteristic function satisfies the equation (where for now we stay with the scale $s_0 = k_1^2$)

$$\chi(\gamma) = \frac{\omega}{\bar{\alpha}_\mathrm{s}} = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma+\omega)\,. \tag{2.2}$$

In the limit $\bar{\alpha}_\mathrm{s} \to 0$ it explicitly reproduces the usual LL characteristic function, (1.2). In the DGLAP limit, $k_1^2 \gg k_2^2$, corresponding to the region close to $\gamma = 0$, one has only the pole $1/\gamma$ — there are no double logarithms, as is required with the scale choice $s_0 = k_1^2$. In the opposite DGLAP limit, $k_2^2 \gg k_1^2$, the relevant region of $\gamma$ is close to $\gamma = 1$. Making the transformation to the relevant scale, $s_0 = k_2^2$, using $\gamma \to \gamma + \omega$, there is only the pole $1/(1-\gamma)$. So one is completely free of double transverse logarithms in both DGLAP limits.

In practice it will be most convenient to work in terms of the symmetric scale choice, $s_0 = k_1 k_2$, with which (2.2) becomes

$$\chi(\gamma) = \frac{\omega}{\bar{\alpha}_\mathrm{s}} = 2\psi(1) - \psi(\gamma + \tfrac{1}{2}\omega) - \psi(1-\gamma+\tfrac{1}{2}\omega)\,. \tag{2.3}$$

Iterating, it is straightforward to obtain the higher order corrections, $\chi_n$, (note the difference in normalisation compared to [3, 4])

$$\chi(\gamma) = \sum_{n=0} \bar{\alpha}_\mathrm{s}^n \chi_n(\gamma)\,.$$

The first two are

$$\chi_1 = -\frac{1}{2}\chi_0(\gamma)\left[\psi'(\gamma) + \psi'(1-\gamma)\right], \tag{2.4}$$

$$\chi_2 = -\frac{1}{2}\chi_1(\gamma)\left[\psi'(\gamma) + \psi'(1-\gamma)\right] - \frac{1}{8}\chi_0(\gamma)^2\left[\psi''(\gamma) + \psi''(1-\gamma)\right]. \tag{2.5}$$

While $\chi_1$ does not correspond exactly to any particular piece of the full NLL kernel, it does reproduce the most divergent $(1/\gamma^3)$ part of the kernel in the regions of $\gamma$ close to

[1] Note that the equation describing the actual LDC model, as used for example in the LDC Monte Carlo event generator [10], differs from BFKL at LL order, as discussed in [9].

0 and close to 1:

$$|\gamma| \ll 1: \quad \chi_1 = -\frac{1}{2\gamma^3} - \frac{\pi^2}{6\gamma} - \zeta(3) + \mathcal{O}(\gamma) , \tag{2.6a}$$

$$|1-\gamma| \ll 1: \quad \chi_1 = -\frac{1}{2(1-\gamma)^3} - \frac{\pi^2}{6(1-\gamma)} - \zeta(3) + \mathcal{O}(1-\gamma) . \tag{2.6b}$$

In fact, up to $\mathcal{O}(1/\gamma)$ it has the same expansion as the following part of the kernel (in the notation of Fadin and Lipatov [3]):

$$-4\left(4\phi(\gamma) - \psi''(\gamma) - \psi''(1-\gamma)\right) ,$$

though this may well be of little significance.

### 2.1. Double-logarithmic and resummed structure

To study the leading double-logarithmic structure of the kernel it suffices to examine the behaviour related just to the pole at $\gamma = 0$, i.e. a "left-hand piece," $\chi_L$, of the characteristic function,

$$\chi_L(\gamma) = \frac{1}{\gamma + \frac{1}{2}\bar{\alpha}_\mathrm{s}\chi_L} . \tag{2.7}$$

The remaining constant piece (including the part coming from $\psi(1-\gamma+\frac{1}{2}\omega)$) has no effect on the leading $\gamma = 0$ DL structure. The solution of (2.7) is

$$\chi_L(\gamma) = \frac{-\gamma \pm \sqrt{\gamma^2 + 2\bar{\alpha}_\mathrm{s}}}{\bar{\alpha}_\mathrm{s}} . \tag{2.8}$$

It has different expansions according to the sign of $\gamma$:

$$\gamma > 0: \quad \chi_L = \frac{1}{\gamma} + \sum_{n=1}^{\infty} d_{n,2n+1} \frac{\bar{\alpha}_\mathrm{s}^n}{\gamma^{2n+1}} , \tag{2.9a}$$

$$\gamma < 0: \quad \chi_L = -\frac{2\gamma}{\bar{\alpha}_\mathrm{s}} - \frac{1}{\gamma} - \sum_{n=1}^{\infty} d_{n,2n+1} \frac{\bar{\alpha}_\mathrm{s}^n}{\gamma^{2n+1}} , \tag{2.9b}$$

where

$$d_{n,2n+1} = (-1)^n \frac{(2n-1)!}{2^{n-1}\,(n-1)!\,(n+1)!} . \tag{2.10}$$

Performing an expansion around $\gamma = 1$ one finds an analogous series in $(1-\gamma)$. It should be emphasised that the result (2.9a) for the leading double logarithmic structure is not in any way model-dependent. It depends only on the fact that with the scale $s_0 = k_1^2$, the kernel should be free of double logarithms around $\gamma = 0$, and that the leading (in $\alpha_\mathrm{s}$) divergence there is $1/\gamma$.



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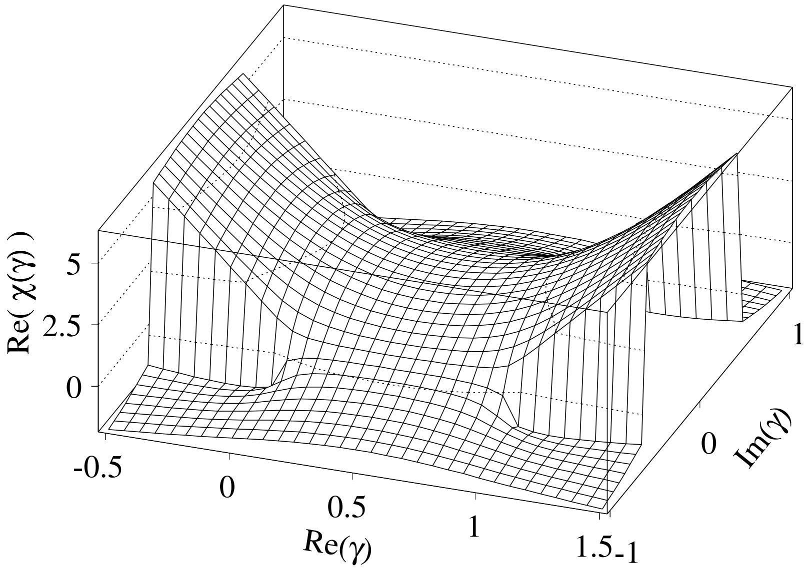


**Figure 1:** The real part of the toy kernel in the complex plane, shown for $\bar{\alpha}_s = 0.2$.

Potentially more model dependent is the behaviour in the regions $\gamma$ and $(1-\gamma)$ very negative, where one sees[2] $\chi \simeq -2\gamma/\bar{\alpha}_s$ and $\chi \simeq -2(1-\gamma)/\bar{\alpha}_s$ respectively. The solution of (2.3), shown in Fig. 1 illustrates this behaviour. The branch points correspond (at least qualitatively) to those in (2.8). As $\bar{\alpha}_s$ goes to zero, they move closer to $\gamma = 0$ and to $\gamma = 1$, while the sections in between them grow steeper, increasing the all-order kernel's similarity to the shape which is familiar in the LL limit (i.e. poles at $\gamma = 0$ and $\gamma = 1$).

## 2.2. Convergence of the toy kernel

The toy kernel is interesting also from the point of view of its convergence properties. Fig. 2 shows the structure of the kernel along the line $\gamma = 1/2 + i\nu$ at various orders for $\bar{\alpha}_s = 0.2$. As in the full NLL kernel [6], there are two symmetric saddle-points at non-zero $\nu$ at NLL order. But if one adds in the NNLL terms one sees that they disappear. At NNNLL (not shown) they come back again. One finds in general that for even orders in $\bar{\alpha}_s$ there is a single saddle-point at $\nu = 0$, while for odd orders in $\bar{\alpha}_s$ the kernel has two symmetric saddle-points at non-zero $\nu$. Fig. 2 shows also the all-order resummed result (a slice of Fig. 1), i.e. the direct solution of (2.3): it has a single saddle point at $\nu = 0$.

This seems to indicate that the splitting into two of the saddle-point in the NLL BFKL kernel, and the associated oscillating solutions are very much artifacts of the truncation of the kernel at finite order (though it turns out, at least in the toy model, that at NLL order the value of $\chi$ at the two saddle-points is a somewhat better approximation to the resummed $\chi(\frac{1}{2})$, than the NLL $\chi(\frac{1}{2})$).

[2]Analogous behaviour has also been observed [11] in the context of the CCFM equation [12], though the slope is different.



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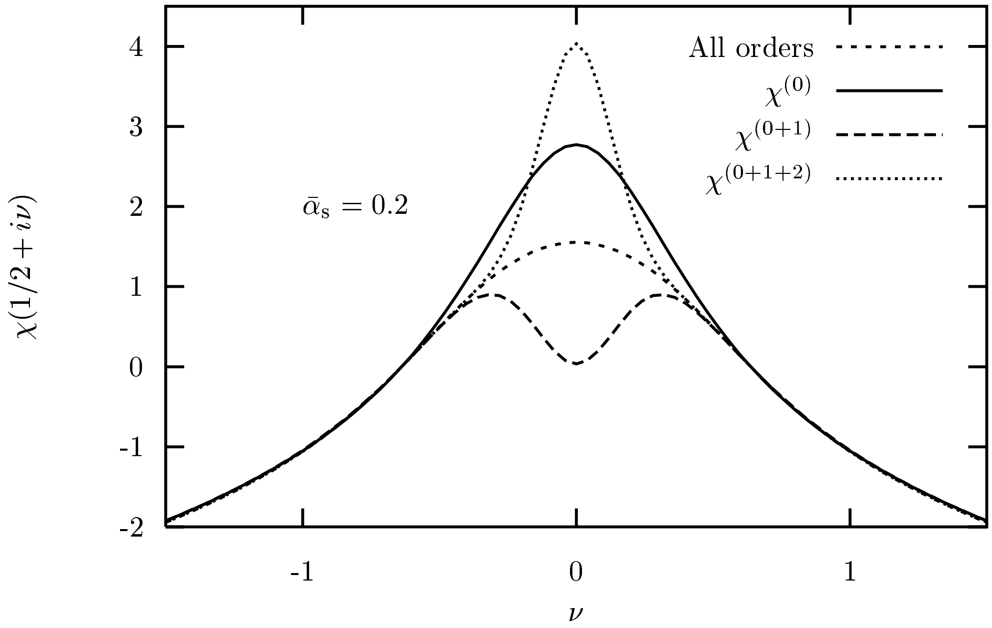


**Figure 2:** The toy kernel along the imaginary axis at various orders. Also shown is the resummed result.

To further study the poor convergence of the toy kernel, one can examine the following ratios,

$$R_p(n) = -\left.\frac{d^p\chi_n/d\gamma^p}{d^p\chi_{n-1}/d\gamma^p}\right|_{\gamma=1/2}, \tag{2.11}$$

as measures of the convergence of the expansion of $\chi$ and its derivatives at $\gamma = \frac{1}{2}$. They are plotted as a function of $n$ in Fig. 3.

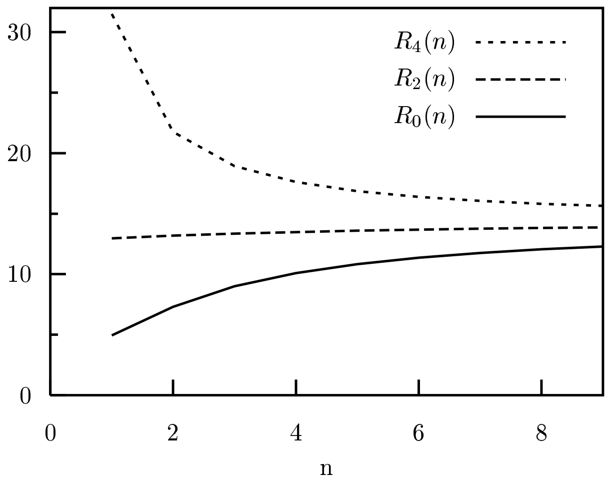


**Figure 3:** $R_0(n)$, $R_2(n)$, $R_4(n)$ as a function of the order of $\alpha_s$, $n$.

At small orders, the convergence is far worse for the derivatives of $\chi$ than for $\chi$ itself. Hence at NLL order, $\chi''(\frac{1}{2})$ goes through zero (i.e. the saddle-point at $\nu = 0$ splits into two) long before $\chi(\frac{1}{2})$ itself does. A simple explanation for the relatively poor convergence of $\chi''(\frac{1}{2})$ compared to that of $\chi(\frac{1}{2})$ is to be had by noting that if the NLL corrections are dominated by a part

$$\chi_1 \simeq \frac{1}{\gamma^3} + \frac{1}{(1-\gamma)^3},$$

then $\chi_1''(\frac{1}{2})/\chi_1(\frac{1}{2}) = 48$, while a similar approximation for the LL kernel gives a relation $\chi_0''(\frac{1}{2})/\chi_0(\frac{1}{2}) = 8$. The difference comes entirely from the $3 \times 4$ obtained in differentiating $1/\gamma^3$ as opposed to $1 \times 2$ in differentiating $1/\gamma$.

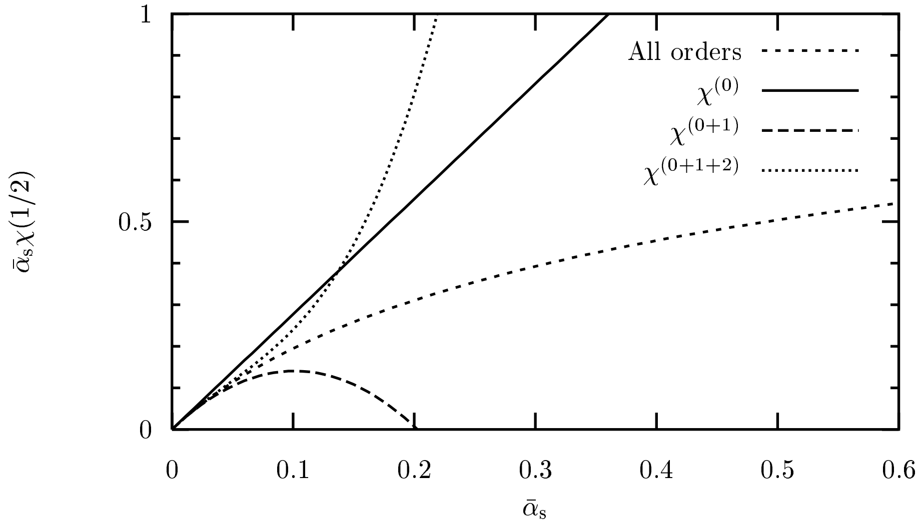


**Figure 4:** The toy-kernel asymptotic power as a function of $\bar{\alpha}_s$, at leading, NLL, NNLL and all orders.

Nevertheless, the perturbative expansion is rather poor even for $\chi(1/2)$, as can be seen from Fig. 4 which shows $\bar{\alpha}_s\chi(\frac{1}{2})$ as a function of $\bar{\alpha}_s$ at different orders. For realistic values of $\bar{\alpha}_s$, say 0.2, the fixed-order expansions are quite unreliable — and going from NLL to NNLL does not bring one any closer to the all-order result except for $\alpha_s \lesssim 0.1$.

To summarise, what one learns from this toy kernel is that the double transverse logarithms lead to a very poor convergence of the BFKL kernel at subleading orders. The features at NLL order (the poor convergence of $\chi(\frac{1}{2})$, the worse convergence of the second derivative, leading to the splitting in two of the saddle-point) are very similar to those observed in the full NLL kernel. But they are quite unrelated to the characteristics (both the analytic structure and the value of $\chi(\frac{1}{2})$) observed after a resummation. Thus the importance of a consistent resummation in the case of the full NLL kernel.

## 3. Improving the full NLL kernel

### 3.1. A resummation procedure

Schematically, the strategy for the construction of a kernel with the appropriate set of double logarithms will consist in the replacement of divergences $\alpha_s^n/\gamma^k$ and $\alpha_s^n/(1-\gamma)^k$ with terms $\alpha_s^n/(\gamma+\frac{1}{2}\omega)^k$ and $\alpha_s^n/(1-\gamma+\frac{1}{2}\omega)^k$, while maintaining the correct expansion to $\mathcal{O}(\alpha_s^n)$.

To understand how to do so systematically, for a kernel known to arbitrary order (the explicit results for NLL order are given in the appendix), let us start with a discussion of the relation between the pattern of divergences in a kernel with $s_0 = k_1^2$, $\xi(\gamma)$ and one with $s_0 = k_1 k_2$, $\chi(\gamma)$, which are related through $\chi(\gamma) = \xi(\gamma + \frac{1}{2}\omega)$ with

$\omega = \bar{\alpha}_s\chi(\gamma)$. We will make the assumption that $\xi(\gamma)$ has an expansion of the form

$$\xi(\gamma) = \sum_{n=0} \sum_{k=-\infty}^{n+1} C_{n,k} \frac{\bar{\alpha}_s^n}{\gamma^k}$$

around $\gamma = 0$. The upper limit on $k$ embodies the statement that $\xi(\gamma)$ is free of double logarithms. If we substitute $\gamma \to \gamma + \frac{1}{2}\omega$ to obtain $\chi(\gamma)$, we immediately see that $\chi(\gamma)$ will contain a set of terms $\bar{\alpha}_s^n \gamma^{-2n+1}$ arising from

$$\frac{1}{\gamma} \longrightarrow \frac{1}{\gamma + \frac{\bar{\alpha}_s}{2\gamma}},$$

(where we have put in $C_{0,1} = 1$). These terms belong to the set resummed in (2.9a). Substituting them into the most divergent term of the NLL kernel, $\alpha_s/\gamma^2$,

$$C_{1,2}\frac{\bar{\alpha}_s}{\gamma^2} \longrightarrow C_{1,2}\frac{\bar{\alpha}_s}{(\gamma + \mathcal{O}\left(\alpha_s^{n+1}/\gamma^{2n+1}\right))^2},$$

and expanding, leads to a set of sub-leading DLs:

$$\frac{\alpha_s}{\gamma^2}\frac{\alpha_s^{n+1}}{\gamma^{2(n+1)}}.$$

together with less leading DLs. More generally the substitution,

$$C_{n,k}\frac{\bar{\alpha}_s^n}{\gamma^k} \longrightarrow C_{n,k}\frac{\bar{\alpha}_s^n}{(\gamma + \frac{1}{2}\bar{\alpha}_s\chi)^k}$$

will lead to a set of terms in $\chi(\gamma)$ of the form

$$\frac{\bar{\alpha}_s^n}{\gamma^k}\left(\frac{\alpha_s}{\gamma^2}\right)^{m-n} = \frac{\bar{\alpha}_s^m}{\gamma^{2(m-n)+k}}$$

for $m \geq n$, together with less divergent terms. So if one has the perturbative series for $\xi(\gamma)$ up to order $N$, in shifting to get $\chi(\gamma)$ one will unambiguously reproduce all double logarithms which are more divergent than those produced by shifting the most divergent unknown term in $\xi$, i.e. $\bar{\alpha}_s^{N+1}\gamma^{-N-2}$, which would give, in $\chi(\gamma)$, terms of the form $\bar{\alpha}_s^n\gamma^{-(2n-N)}$, for $n > N$. In other words one correctly reproduces in $\chi(\gamma)$ all terms of $\mathcal{O}\left(\alpha_s^n\right)$ for $n \leq N$, and additionally double-logarithmic terms $\mathcal{O}\left(\bar{\alpha}_s^n\gamma^{-k}\right)$ for $2n+1-N \leq k \leq 2n+1$, and $n > N$.

Hence if we can construct a kernel which has no double logarithms, $\alpha_s^n\gamma^{-(n+1+k)}$, $k > 0$, for a scale $s_0 = k_1^2$, no DLs, $\alpha_s^n(1-\gamma)^{-(n+1+k)}$, for a scale $s_0 = k_2^2$, and which is exact to order $N$, we will "for free" correctly obtain, for scale $s_0 = k_1k_2$ all DLs, $\bar{\alpha}_s^n\gamma^{-k}$ and $\bar{\alpha}_s^n(1-\gamma)^{-k}$ with $2n+1-N \leq k \leq 2n+1$. More specifically, for the NLL case the aim is to construct a kernel whose most divergent terms for $s_0 = k_1^2$ are $\alpha_s^n/\gamma^{n+1}$, and analogously for scale $s_0 = k_2^2$ — modifying the most divergent unknown term $\alpha_s^2/\gamma^3$ can then at worst change the sub-sub-leading double logarithms, $\alpha_s^n/\gamma^{2n-1}$, $n \geq 2$. In other words, the leading and sub-leading double logarithms will be known unambiguously

So returning to the general case, let our resummed kernel be denoted by $\chi^{(N)}(\gamma)$ (for scale $s_0 = k_1 k_2$), with a perturbative expansion

$$\chi^{(N)}(\gamma) = \sum_{n=0}^{N} \bar{\alpha}_s^n \chi_n(\gamma) + \sum_{n=N+1}^{\infty} \bar{\alpha}_s^n \chi_n^{(N)}(\gamma)\,,$$

where $\chi_n$ is the exact order $n$ contribution to the BFKL kernel.

For each order $n \geq 0$ of the exact kernel, $\chi_n(\gamma)$, there will be divergences $d_{n,k}/\gamma^k$, for $1 \leq k \leq 2n+1$, with $d_{n,k}$ being numerical coefficients. There will be similar divergences at $\gamma = 1$ with coefficients $\bar{d}_{n,k}$. Furthermore, $\chi_n^{(N)}(\gamma)$ will have divergences $\bar{\alpha}_s^n/\gamma^k$ with coefficients $d_{n,k}^{(N)}$ (and similarly for $(1-\gamma)$). These coefficients will have the property

$$d_{n,k}^{(N)} = d_{n,k} \quad \text{for} \quad (2n+1-N \leq k \leq 2n+1)\,.$$

To construct a kernel $\chi^{(N)}$, it is useful to specify a set of functions $D_k(\gamma)$ which are regular for $\gamma > 1/2$, and at $\gamma = 0$ have only the divergence $1/\gamma^k$. We can then immediately construct a kernel $\chi^{(0)}(\gamma)$ which is free of DLs in the appropriate limits and which is exact to $\mathcal{O}(\bar{\alpha}_s^0)$:

$$\begin{aligned}\chi^{(0)}(\gamma) = \chi^{(0)}(\gamma, \omega = \bar{\alpha}_s \chi^{(0)}) = \\ \chi_0(\gamma) + d_{0,1}[D_1(\gamma + \tfrac{1}{2}\omega) - D_1(\gamma)] + \bar{d}_{0,1}[D_1(1-\gamma+\tfrac{1}{2}\omega) - D_1(1-\gamma)]. \quad (3.1)\end{aligned}$$

To go to $N > 0$ one uses the recursion relation

$$\begin{aligned}\chi^{(N)}(\gamma) = \chi^{(N)}\left(\gamma, \omega = \bar{\alpha}_s \chi^{(N)}\right) = \chi^{(N-1)}(\gamma, \omega) + \bar{\alpha}_s^N \left(\chi_N(\gamma) - \chi_N^{(N-1)}(\gamma)\right) + \\ + \bar{\alpha}_s^N \sum_{k=1}^{N+1} \Big[\left(d_{N,k} - d_{N,k}^{(N-1)}\right)[D_k(\gamma + \tfrac{1}{2}\omega) - D_k(\gamma)] \\ + \left(\bar{d}_{N,k} - \bar{d}_{N,k}^{(N-1)}\right)[D_k(1-\gamma+\tfrac{1}{2}\omega) - D_k(1-\gamma)]\Big]. \quad (3.2)\end{aligned}$$

To demonstrate that $\chi^{(N)}$ correctly reproduces the kernel up to order $\bar{\alpha}_s^N$ it suffices to note that neither the parts in square brackets, nor

$$\chi^{(N-1)}(\gamma, \omega) - \bar{\alpha}_s^N \chi_N^{(N-1)}(\gamma)$$

contains any terms at $\mathcal{O}\left(\bar{\alpha}_s^N\right)$, so that the entire $\mathcal{O}\left(\bar{\alpha}_s^N\right)$ part comes from $\chi_N(\gamma)$.

The next step is to show that the characteristic function written in terms of the scale $s_0 = k_1^2$, $\xi(\gamma) = \chi(\gamma - \frac{1}{2}\omega)$ is free of DL divergences, $\bar{\alpha}_s^n \gamma^{-k}$ for $k > n+1$. It is convenient to rewrite the kernel as:

$$\begin{aligned}\chi^{(N)}(\gamma) = \chi^{(N)}\left(\gamma, \omega = \bar{\alpha}_s \chi^{(N)}\right) = \sum_{n=0}^{N} \bar{\alpha}_s^n R_n(\gamma) + \\ + \sum_{n=0}^{N} \bar{\alpha}_s^n \sum_{k=1}^{n+1} \Big[\left(d_{n,k} - d_{n,k}^{(n-1)}\right) D_k(\gamma + \tfrac{1}{2}\omega) + \left(\bar{d}_{n,k} - \bar{d}_{n,k}^{(n-1)}\right) D_k(1-\gamma+\tfrac{1}{2}\omega)\Big], \quad (3.3)\end{aligned}$$

where

$$R_n(\gamma) = \chi_n(\gamma) - \chi_n^{(n-1)}(\gamma) -$$
$$- \sum_{k=1}^{n+1} \left[ \left( d_{n,k} - d_{n,k}^{(n-1)} \right) D_k(\gamma) + \left( \bar{d}_{n,k} - \bar{d}_{n,k}^{(n-1)} \right) D_k(1-\gamma) \right], \qquad (3.4)$$

with $\chi^{(-1)} = 0$ and all $d_{n,k}^{(-1)} = 0$. The fact that $\chi_n^{(n-1)}(\gamma)$ has the same pattern of $\gamma^{-k}$ and $(1-\gamma)^{-k}$, $k > n+1$, divergences as $\chi_n(\gamma)$ means that $R_n(\gamma)$ is regular, for each $n$, at $\gamma = 0$ and $\gamma = 1$. Applying the transformation $\gamma \to \gamma - \frac{1}{2}\omega$ to get $\xi(\gamma)$ gives us

$$\xi^{(N)}(\gamma) = \xi^{(N)}\left(\gamma, \omega = \bar{\alpha}_s \, \xi^{(N)}\right) = \sum_{n=0}^{N} \bar{\alpha}_s^n R_n(\gamma - \tfrac{1}{2}\omega) + \qquad (3.5)$$
$$+ \sum_{n=0}^{N} \bar{\alpha}_s^n \sum_{k=1}^{n+1} \left[ \left( d_{n,k} - d_{n,k}^{(n-1)} \right) D_k(\gamma) + \left( \bar{d}_{n,k} - \bar{d}_{n,k}^{(n-1)} \right) D_k(1-\gamma+\omega) \right]. \qquad (3.6)$$

The first half of the second line produces a set of divergences $\bar{\alpha}_s^n \gamma^{-k}$, with $k \le n+1$; i.e. that part is free of double logarithms. Substituting these into the second half of the second line and into $R_n$ will at worst produce extra divergences $\bar{\alpha}_s^n \gamma^{-n}$. Hence $\xi(\gamma)$ is free of double logarithmic divergences, and exact to $\mathcal{O}(\bar{\alpha}_s^n)$ (since $\chi^{(N)}$ was exact to $\mathcal{O}(\bar{\alpha}_s^n)$). Similarly changing scale to $s_0 = k_2^2$, there will be no DL divergences at $\gamma = 1$.

This completes the demonstration that $\chi^{(N)}$ is exact up to $\mathcal{O}(\bar{\alpha}_s^n)$ and that it correctly reproduces the double logarithms $\bar{\alpha}_s^n \gamma^{-k}$ and $\bar{\alpha}_s^n (1-\gamma)^{-k}$ with $2n+1-N \le k \le 2n+1$.

Analytically the expression for the series of the leading double logarithms has been given in (2.9a, 2.10). The sub-leading double logarithms, $\bar{\alpha}_s^n/\gamma^{2n}$, are determined entirely by the coefficient of $\bar{\alpha}_s/\gamma^2$, $d_{1,2}$ (which itself depends on one's choice of scale for $\alpha_s$):

$$d_{n,2n} = (-1)^{n+1} \, d_{1,2} \, \frac{(2n-1)!}{n! \, (n-1)! \, 2^{n-1}} \, .$$

### 3.2. Resummation schemes

The procedure described above is not unique in that there is some freedom in the choice of the divergent functions $D_k(\gamma)$. This freedom results in differences in the non-guaranteed terms of $\chi^{(N)}$, and is in some sense a measure of the uncertainties due to unknown higher order terms.

To study the properties of the resummed NLL kernel, and the ambiguities which remain, we will consider four different choices for the treatment of uncontrolled sub-leading terms. Extra details (including the explicit forms for schemes 1 and 3) are to be found in the appendix.

1. A choice similar to that used in the toy model:

$$D_k(\gamma) = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{d\gamma^{k-1}} \left[ \psi(1) - \psi(\gamma) \right] .$$

2. Even simpler, is

$$D_k(\gamma) = \frac{1}{\gamma^k}\,.$$

3. Both of these choices will run into problems at some value of $\alpha_s$, because even after resumming double-logarithmic contributions there are terms such as $\bar\alpha_s^n/\gamma^{n+1}$ which can be dangerous. Their resummation is related to the reproduction of the correct DGLAP limit and a resummation of the running of $\bar\alpha_s$. Both should not be difficult to do, but for simplicity will not be done exactly. We will simply modify the resummation procedure suggested above and for $\chi^{(0)}$ use

$$\begin{aligned}\chi^{(0)}(\gamma) &= \chi^{(0)}(\gamma, \omega = \bar\alpha_s\chi^{(0)}) = \\ &= (1-\bar\alpha_s A)\left[2\psi(1) - \psi(\gamma + \tfrac{1}{2}\omega + \bar\alpha_s B) - \psi(1-\gamma+\tfrac{1}{2}\omega + \bar\alpha_s B)\right]\,, \end{aligned} \tag{3.7}$$

with the constants $A$ and $B$ chosen such that $d_{1,k}^{(0)} = d_{1,k}$ for $k \geq 1$, not just for $k=3$. The choice of the $D_k$ is then immaterial since they are not needed at $\mathcal{O}(\alpha_s)$.

4. A similar procedure (but more in line with scheme 2) is to use

$$\begin{aligned}\chi^{(0)}(\gamma) = \chi^{(0)}(\gamma, \omega = \bar\alpha_s\chi^{(0)}) &= \chi_0(\gamma) - \frac{1}{\gamma} - \frac{1}{1-\gamma} + \\ &+ (1-\bar\alpha_s A)\left[\frac{1}{\gamma+\frac{1}{2}\omega+\bar\alpha_s B} + \frac{1}{1-\gamma+\frac{1}{2}\omega+\bar\alpha_s B}\right], \end{aligned} \tag{3.8}$$

where again $A$ and $B$ are chosen such that $d_{1,k}^{(0)} = d_{1,k}$ for $k \geq 1$.

Results using these four resummation schemes are shown in Fig. 5. The value of $\bar\alpha_s\chi(\frac{1}{2})$ turns out to be relatively stable under changes of scheme — and quite different from the pure NLL result. Indeed it seems somewhat more in line with what one might expect from phenomenology. The reason for the larger difference between scheme 4 and the other schemes is not entirely clear.

The results for $\bar\alpha_s\chi''(\frac{1}{2})$ show quite a considerable spread. This is connected with the treatment of the $\bar\alpha_s/\gamma^2$ and $\bar\alpha_s/(1-\gamma)^2$ terms, which remain "untamed" even after the resummation of the $\alpha_s^n\gamma^{-(2n+1)}$ and $\alpha_s^n\gamma^{-2n}$ terms. In practice they have quite a large coefficient and precisely because they are $\propto 1/\gamma^2$ their contribution to the second derivative is enhanced (by a factor of 3 naively) as compared to their contribution to $\chi(\frac{1}{2})$. Once the $\alpha_s^n\gamma^{-(n+1)}$ terms are resummed (which fixes also the DL terms $\alpha_s^n\gamma^{-(2n-1)}$), as in schemes 3 and 4 (which resum $\alpha_s^n\gamma^{-(n+1)}$ terms in the same way), one finds a certain independence on the exact resummation scheme used for the DLs. It should be emphasised however that the scheme used here for resumming the $\alpha_s^n\gamma^{-n}$ terms has been chosen *ad hoc*, so that all that should be taken seriously is the qualitative effect of a sufficient resummation, namely that it ensures that the second derivative stays positive, and that the saddle point at $\gamma = \frac{1}{2}$ does not split into two.



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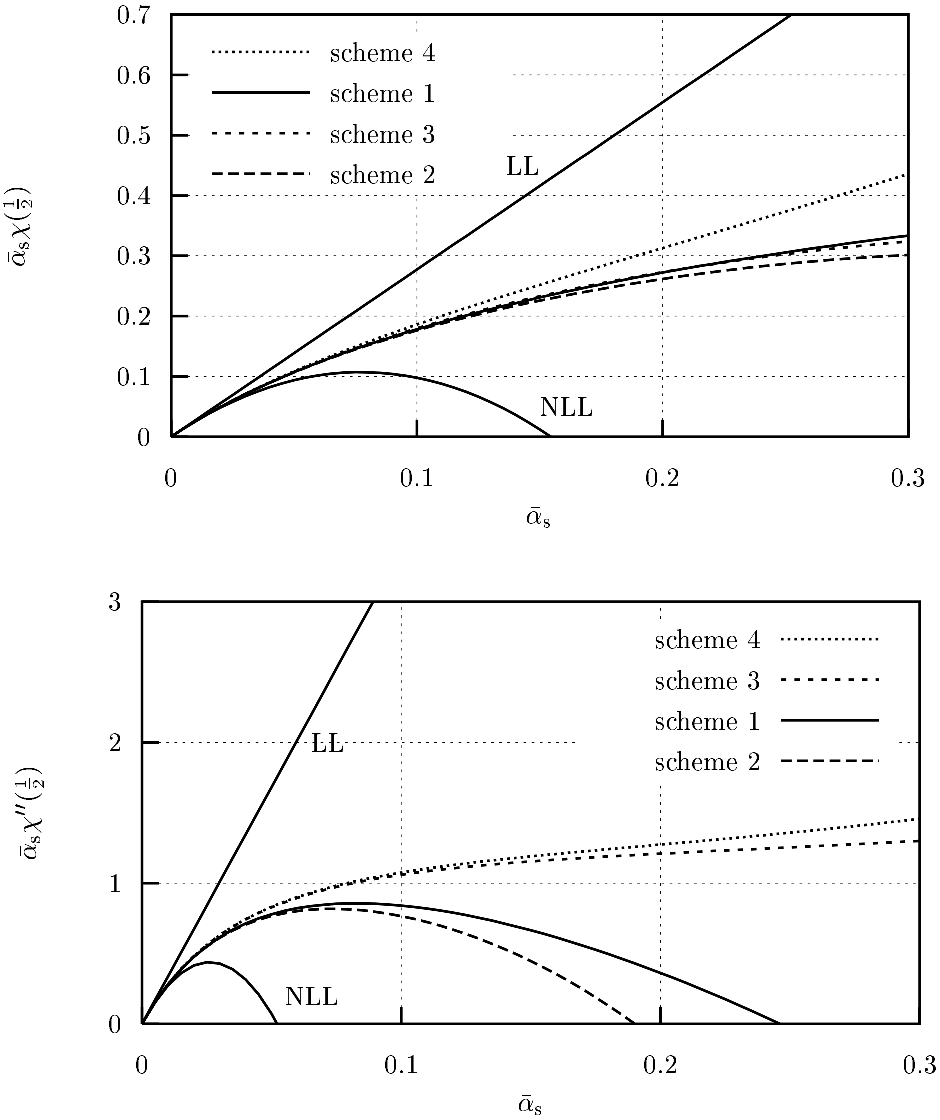


**Figure 5:** The result of resummation on $\chi$ and its second derivative; shown for $n_f = 0$. A symmetric version of the pure NLL kernel has been used, corresponding to the suggestion [3] to redefine $k^{2(\gamma-1)}$ as $\sqrt{\alpha_s(k^2)}k^{2(\gamma-1)}$ in the eigenfunctions.

## 4. Discussion

The main message of this paper is that the pathological aspects of the NLL BFKL kernel have to be, and can be cured by suitable resummations.[3] A first step along the way is to resum leading and sub-leading double logarithms, which modifies the NLL kernel as shown in Fig. 5.

[3] This differs from the conclusions presented in [5], where it is suggested that the calculation of the NNLL and NNNLL-order terms would suffice to improve the convergence. The origin of the difference in conclusions lies perhaps in the fact that the authors of [5] study predominantly the anomalous dimension (the expansion around $\gamma = 0$, which is less sensitive to double-logarithms from $\gamma = 1$) while the aim of this paper is to examine the properties of the kernel around $\gamma = 1/2$.

The approach used here for determining the double logarithms relies on two elements: the absence of double transverse logarithms in the DGLAP limits (with scale choices $k_1^2$ or $k_2^2$) and the assumption that one can change the scale of the kernel consistently at all orders by a transformation of the form

$$\gamma \to \gamma \pm \tfrac{1}{2}\bar{\alpha}_{\rm s}\chi(\gamma)\,.$$

This is tied in with the assumption that the kernel $\bar{\alpha}_{\rm s}\chi(\gamma)$ exponentiates, i.e. that one can write the gluon's Green function as in (1.1). At NNLL order, one knows that exponentiation no longer holds, due for example to the running of $\alpha_{\rm s}$ [13, 7] and unitarity corrections, so that a change of scale is no longer accomplished by a simple shift of $\gamma$. To try to understand the effect of changing the scale in a kernel which fails to exponentiate at NNLL order, one can take as a rough example, inspired by [13, 7], a worst-case situation where the non-exponentiation results in

$$\exp\left(\frac{\alpha_{\rm s}\ln s}{\gamma}\right) \to \exp\left(\frac{\alpha_{\rm s}\ln s}{\gamma} + \mathcal{O}\left(\frac{\alpha_{\rm s}^5\ln^3 s}{\gamma^5}\right)\right).$$

Working through the effect of a change in scale of $s$, one finds that the shift in $\gamma$ breaks down at a level which can affect terms $\alpha_{\rm s}^n/\gamma^{2n-1}$ in the $s_0 = k_1k_2$ kernel. This suggests that the leading and sub-leading DLs worked out in this paper should not be affected by the non-exponentiation of the kernel.

The resummation of double logarithms is probably sufficient to obtain a reasonable idea of the typical power that one may expect at small $x$. But to carry out serious phenomenology it is vital to have adequate control over the value of the second derivative, $\chi''(\frac{1}{2})$. This requires the additional resummation of $\alpha_{\rm s}^n\gamma^{-(n+1)}$ terms. A qualitative illustration of the effect of such a resummation is given with schemes 3 and 4 in Fig. 5. A correct treatment would involve a resummation of collinear logarithms (which perhaps can be accomplished [14] simply by shifting the $1/\gamma$ to $1/(\gamma + \mathcal{O}(\alpha_{\rm s}))$ as done in schemes 3 and 4) and additionally the resummation of logarithms associated with the running of $\alpha_{\rm s}$, which will behave differently: for example evaluating the kernel with $\alpha_{\rm s}$-scale $k_1$, close to $\gamma = 1$, accounting just for running coupling effects in a single emission, one will have

$$\begin{aligned}\alpha_{\rm s}(k_1)\chi(\gamma) &\sim \bar{\alpha}_{\rm s}(k_1^2)\int_{k_1^2}^{\infty}\frac{dk_2^2}{k_2^2}\left(\frac{k_2^2}{k_1^2}\right)^{\gamma-1}\frac{1}{1+\bar{\alpha}_{\rm s}(k_1^2)\,\beta_0\ln k_2^2/k_1^2}\\ &= \bar{\alpha}_{\rm s}(k_1^2)\sum_{n=0}^{\infty}(-1)^n n!\frac{(\bar{\alpha}_{\rm s}\beta_0)^n}{(1-\gamma)^{n+1}}\,.\end{aligned}$$

which sums to an incomplete $\Gamma$-function, and has a distinctly different structure from the resummation of collinear logarithms — so there remains the job of putting them together. There are of course many other issues associated with the running of $\alpha_{\rm s}$. These have already been considered in some detail by a number of groups [13, 7, 15]. Finally it might be possible to understand, at all orders, contributions which are enhanced neither at $\gamma = 0$ nor at $\gamma = 1$ but which are still important at $\gamma = \frac{1}{2}$, such as those arising from angular ordering [12, 16].

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## A. Application of the resummation to the NLL kernel

For completeness, it is useful to discuss the application to the NLL kernel of the resummation procedure discussed in section 3.1. It will be illustrated explicitly for the cases of schemes 1 and 3, using a symmetric version (corresponding to the extraction of $\bar{\alpha}_s(k_1k_2)$, or $\sqrt{\bar{\alpha}_s(k_1^2)\bar{\alpha}_s(k_2^2)}$ ) of the NLL kernel:

$$\begin{aligned}4\chi_1(\gamma) = -\Bigg[&\frac{2\beta_0}{C_A}\chi_0^2(\gamma) - K\chi_0(\gamma) - 6\zeta(3) + \\ &+\frac{\pi^2\cos(\pi\gamma)}{\sin^2(\pi\gamma)(1-2\gamma)}\left(3+\left(1+\frac{n_f}{C_A^3}\right)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right) - \\ &-\psi''(\gamma) - \psi''(1-\gamma) - \frac{\pi^3}{\sin(\pi\gamma)} + 4\phi(\gamma)\Bigg], \end{aligned} \tag{A.1}$$

where

$$\begin{aligned}\beta_0 &= \frac{11C_A}{12} - \frac{2n_f}{12}, \\ K &= \frac{67}{9} - \frac{\pi^2}{3} - \frac{10n_f}{9C_A},\end{aligned}$$

and

$$\phi(\gamma) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{\psi(n+1+\gamma)-\psi(1)}{(n+\gamma)^2} + \frac{\psi(n+2-\gamma)-\psi(1)}{(n+1-\gamma)^2}\right].$$

First we must determine the pattern of divergences $\alpha_s^n/\gamma^k$ ($\bar{d}_{n,k} = d_{n,k}$) of the LL and NLL kernels:[4]

$$\begin{aligned}d_{0,1} &= 1, \\ d_{1,1} &= -\frac{67}{36} - \frac{13n_f}{36C_A^3} + \frac{\pi^2}{12} + \frac{K}{4}, \\ d_{1,2} &= -\frac{\beta_0}{2C_A} - \frac{11}{12} - \frac{n_f}{6C_A^3}, \\ d_{1,3} &= -\frac{1}{2}.\end{aligned}$$

The steps needed to resum the DLs in the kernel then depend on the scheme used.

[4]One notes that $n_f$-independent part of $d_{1,1}$ is zero.

## A.1. Scheme 1

In scheme 1, the terms used to subtract off the divergences $1/\gamma^k$, $D_k(\gamma)$ are

$$D_1(\gamma) = \psi(1) - \psi(\gamma)\,,$$
$$D_2(\gamma) = \psi'(\gamma)\,.$$

Applying (3.1), the kernel correct to LL order and with all the leading DLs is just the toy-kernel discussed in section 2:

$$\chi^{(0)}(\gamma) = \chi^{(0)}(\gamma, \omega = \bar{\alpha}_{\mathrm{s}}\chi^{(0)}) = 2\psi(1) - \psi(\gamma + \tfrac{1}{2}\omega) - \psi(1 - \gamma + \tfrac{1}{2}\omega)\,.$$

The next step is to determine the NLL part of $\chi^{(0)}$ (to avoid any double-counting when one adds in the pure NLL part of the kernel). It is given by (2.4):

$$\chi_1^{(0)} = -\frac{1}{2}\chi_0(\gamma)[\psi'(\gamma) + \psi'(1-\gamma)]\,,$$

and the coefficients of its divergences ($\bar{d}_{n,k}^{(0)} = d_{n,k}^{(0)}$) are

$$d_{1,1}^{(0)} = -\frac{\pi^2}{6}\,,$$
$$d_{1,2}^{(0)} = 0\,,$$
$$d_{1,3}^{(0)} = -\frac{1}{2}\,.$$

The fact that $d_{1,3}^{(0)} = d_{1,3}$ is a consequence of $\chi^{(0)}$ correctly reproducing the DLs that are present at NLL order. Using (3.2) we are now in a position to write down $\chi^{(1)}$, i.e. the kernel which is correct to NLL order and contains the appropriate leading and sub-leading DLs. Essentially it is obtained by "shifting" the divergences which are not already accounted for by $\chi_1^{(0)}$

$$\begin{aligned}\chi^{(1)}(\gamma) &= \chi^{(0)}\left(\gamma, \omega = \bar{\alpha}_{\mathrm{s}}\chi^{(1)}\right) = \chi^{(0)}(\gamma,\omega) + \bar{\alpha}_{\mathrm{s}}\left(\chi_1(\gamma) - \chi_1^{(0)}\right) \\ &+ \sum_{k=1}^{2} \bar{\alpha}_{\mathrm{s}}(d_{1,k} - d_{1,k}^{(0)})\left(D_k(\gamma + \tfrac{1}{2}\omega) + D_k(1-\gamma+\tfrac{1}{2}\omega) - D_k(\gamma) - D_k(1-\gamma)\right). \end{aligned} \quad \text{(A.4)}$$

This equation applies also to scheme 2, as long as one modifies $\chi^{(0)}$, $\chi_1^{(0)}$ and $d_{1,k}^{(0)}$ appropriately.

## A.2. Scheme 3

Schemes 3 and 4 differ from schemes 1 and 2 in the form used for $\chi^{(0)}$, which is constructed in such a way that $d_{1,k}^{(0)} = d_{1,k}$ not just for $k = 3$, but also for $k = 1, 2$; they resum (in a manner which given the present state of the art is arbitrary) the $\alpha_s^n/\gamma^n$ and $\alpha_s^n/\gamma^{n+1}$ divergences. For scheme 3, where one uses

$$\begin{aligned}\chi^{(0)}(\gamma) = \chi^{(0)}(\gamma, \omega = \bar{\alpha}_s \chi^{(0)}) = \\ (1-\bar{\alpha}_s A)\left[2\psi(1) - \psi(\gamma + \tfrac{1}{2}\omega + \bar{\alpha}_s B) - \psi(1-\gamma+\tfrac{1}{2}\omega + \bar{\alpha}_s B)\right], \quad \text{(A.5)}\end{aligned}$$

the NLL part of $\chi^{(0)}$ is

$$\chi_1^{(0)} = -\left(B + \frac{1}{2}\chi_0(\gamma)\right)[\psi'(\gamma) + \psi'(1-\gamma)] - A\,\chi_0\,,$$

and its divergences are given by

$$\begin{aligned}d_{1,1}^{(0)} &= -A - \frac{\pi^2}{6}\,,\\ d_{1,2}^{(0)} &= -B\,,\\ d_{1,3}^{(0)} &= -\frac{1}{2}\,.\end{aligned}$$

Since we want $\chi_1^{(0)}$ to reproduce the divergences of $\chi_1$, i.e. $d_{1,k}^{(0)} = d_{1,k}$, we have to set $A$ and $B$ as

$$\begin{aligned}A &= d_{1,1} - \frac{\pi^2}{6}\,,\\ B &= d_{1,2}\,.\end{aligned}$$

One notes (by carrying out the shift) that $\chi^{(0)}$ is free of the appropriate double logarithms when one changes scale to $k_1^2$ or $k_2^2$. To obtain the version of the kernel which is correct to NLL order, one then applies a similar procedure to that of (3.2), except that there is no need for any piece analogous to the second line of (A.4), since $d_{1,k}^{(0)} = d_{1,k}$:

$$\chi^{(1)}(\gamma) = \chi^{(0)}\left(\gamma, \omega = \bar{\alpha}_s\chi^{(1)}\right) = \chi^{(0)}(\gamma,\omega) + \bar{\alpha}_s\left(\chi_1(\gamma) - \chi_1^{(0)(\gamma)}\right). \quad \text{(A.8)}$$

Since the second term on the RHS is finite at $\gamma = 0$ and $\gamma = 1$, the fact that $\chi^{(0)}$ is free of the relevant double logarithms when transformed to scales $k_1^2$ or $k_2^2$, is sufficient to guarantee that $\chi^{(1)}$ has this property too, and hence that for scale $k_1 k_2$ it correctly reproduces the leading and sub-leading DLs.

The procedure used for scheme 4 is analogous.

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